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Critical probability and scaling functions of bond percolation on two-dimensional random lattices

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Abstract. We locate the critical probability of bond percolation on two-dimensional random lattices as $p_c = 0.3329(6)$. Because of the symmetry with respect to permutation of the two axes for random lattices, we expect that for an aspect ratio of unity and sufficiently large lattices, the probability of horizontal spanning equals the probability of vertical spanning. This is confirmed by our Monte Carlo simulations. We show that the ideas of universal scaling functions and non-universal metric factors can be extended to random lattices by studying the existence probability E_p and the percolation probability P on finite square, planar triangular, and random lattices with periodic boundary conditions using a histogram Monte Carlo method. Our results also indicate that the metric factors may be the same between random lattices and planar triangular lattices provided that the aspect ratios are 1 and $\sqrt{3}/2$.

Random lattices were first employed by Christ, Friedberg and Lee (CFL) to formulate another type of lattice field theory [1–3]. In these lattices, the volume is partitioned into non-overlapping simplices whose vertices are random-distributed lattice sites, and then the coordination number of each lattice site is randomly distributed with the average number approaching six for two dimensions in the infinite case. Also, by construction, we can have random lattices with either periodic boundary conditions or formed on a hyperspherical surface. An example of the two-dimensional random lattices used in this work and constructed by the CFL algorithm [1] is given in figure 1. In [4], Hsu *et al* have performed Monte Carlo simulations for bond percolation processes on two-dimensional random lattices and their duals, and from the results of scaling powers they concluded that the idea of universality can be extended to random lattices. However, besides the critical exponents, it is also important to understand the scaling behaviour exhibited by finite systems on random lattices. A similar issue was investigated by Espriu *et al* [5] for the two-dimensional Ising model on random lattices to discover whether the specific heat singularity is more logarithmic or log–log. No conclusive result was obtained, but their study showed evidence that the random lattice specific heat agrees with the Onsager result and disagrees with dilute Ising model results. In this paper we extend the work of [4] to investigate the behaviour of scaling functions for the bond percolation processes on random lattices.

The finite-size scaling was first formulated in 1971 by Fisher [6, 7]. Since then it has been widely used for practical purposes. It is an efficient way to extrapolate the data obtained from Monte Carlo simulations on finite systems to the thermodynamic limit. It also serves as the basis of the real space renormalization group method [8]. Let us consider

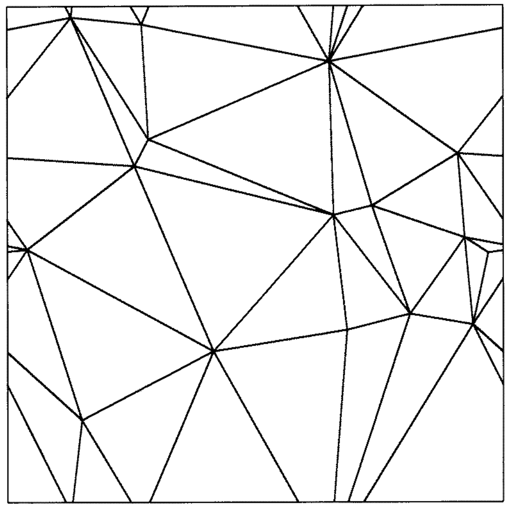

Figure 1. An example of a random lattice on a two-dimensional square area with periodic boundary condition.

some physical quantity X, which is a function of the reduced probability $t \equiv p - p_c$, in percolation processes. According to the finite-size scaling hypothesis, if the quantity X is known to scale as $X(t) \sim t^{-\rho}$ near the critical point $t = 0$ in the infinite system, then we can expect that for a finite system characterized by a size L this quantity $X_L(t)$ should obey the general scaling law,

$$X_L(t) \sim L^{\rho/\nu} F(tL^{1/\nu}) \tag{1}$$

where ν is the correlation length exponent, and $F(x)$ with $x = tL^{1/\nu}$ is called a scaling function. When finite-size scaling is valid, the scaled data $X_L(t)L^{-\rho/\nu}$ for different values of L and t are described by a single scaling function $F(x)$. Privman and Fisher [9] further introduced the concept of universal scaling functions and non-universal metric factors. They argued that the only non-universal factors are the metrical ones relating the relevant variables to the physical parameters, and no other non-universal parameters enter into finite-size scaling formulae. The non-universal metric factors may depend on the shapes and the boundary conditions of the lattices as discussed by Ziff [10] and Ziff *et al* [11]. Specifically, Privman and Fisher proposed that near the critical point $t = 0$, the singular part of a free energy can be written as

$$f_L^s(t) \sim L^{-d} Y(DtL^{1/\nu}) \tag{2}$$

where d is the spatial dimensionality of the lattice, Y is a universal scaling function, and D is a non-universal metric factor. Studies have been performed in a variety of models in this aspect. Henkel computed the finite-size corrections for all energy gaps in the spectrum of the quantum Ising chain in the finite-size scaling limit for periodic and antiperiodic boundary conditions [12], and the scaling functions in the two-dimensional tricritical model [13]. Burkhardt and Guim [14] studied the scaling functions for spin–spin and energy–energy correlations in Ising strips with various boundary conditions. Reinicke [15] and Reinicke and Vescan [16] calculated analytical and non-analytical corrections to finite-size scalings in the Ising and three-state Potts models. These models were also studied by Debierre and Turban [17]. Lee evaluated the scaling function and the non-universal metric factors for the three-state Potts model on the square lattices [18]. Hu, Lin and Chen (HLC) showed that by choosing a very small number of non-universal metric factors all scaled data of the existence probability E_{p} and the percolation probability P of percolation processes on two-dimensional regular lattices may fall on the same universal scaling functions, and free and periodic boundary conditions share the same non-universal metric factors [19, 20]. Hovi and Aharony [21] pointed out that the HLC results confirm the prediction from the renormalization group approach [21]. Okabe and Kikuchi extended the work of HLC to the two-dimensional Ising model on regular lattices and quasiperiodic lattices [22]. One purpose of this work is to investigate what kind of features of universal scaling functions and non-universal factors appear on random lattices. First, we employ the scaling form of equation (1) to give a better estimate of the critical probability p_{c}. Then we investigate the universal scaling functions and the non-universal metric factors for the existence probability of percolation E_{p} and the percolation probability P on finite square, planar triangular, and random lattices with periodic boundary conditions.

In the work of [4], the renormalization group method is used on rather small random lattices to estimate the critical probability p_{c}, and the resultant value is $p_{\mathrm{c}} = 0.3387$. To give a better estimate of p_{c}, we use the scaling form of the existence probability of percolation, $E_{\mathrm{p}}(G(L), p)$, which is defined as the average numbers of percolating clusters on random lattice G of size L for a given link occupied probability p. Note that one can use different spanning rules to define a percolation cluster [23, 24]. In this work, we use $E_{\mathrm{p}}^{\mathrm{v}}$ for the average number of percolating clusters of vertical crossings, $E_{\mathrm{p}}^{\mathrm{h}}$ for those of horizontal crossings, $E_{\mathrm{p}}^{\mathrm{v+h}}$ for the sum of $E_{\mathrm{p}}^{\mathrm{v}}$ and $E_{\mathrm{p}}^{\mathrm{h}}$, $E_{\mathrm{p}}^{\mathrm{v\cap h}}$ for those of simultaneous vertical and horizontal crossings, and E_{p} for the difference between $E_{\mathrm{p}}^{\mathrm{v+h}}$ and $E_{\mathrm{p}}^{\mathrm{v\cap h}}$. Because the critical exponent of E_{p} is zero, the finite-size scaling suggests that $E_{\mathrm{p}}^{\mathrm{v+h}}$ in the vicinity of p_{c} may take the form

$$E_{\mathrm{p}}^{\mathrm{v+h}}(G(L), p) = 1 + A(p - p_{\mathrm{c}})L^{1/\nu} + B \tag{3}$$

where B is of the order of $[(p - p_{\mathrm{c}})L^{1/\nu}]^2$ and $\nu = \frac{4}{3}$. As pointed out by Langlands *et al* [25], this form indicates that one may have two ways to estimate p_{c}. One is to simulate the quantity $E_{\mathrm{p}}^{\mathrm{v+h}}$ as a function of p, and then to calculate its intercept with $E_{\mathrm{p}}^{\mathrm{v+h}} = 1$. The other is to simulate the quantity $E_{\mathrm{p}}^{\mathrm{v+h}}$ as a function of p for two different values of L, and then to find the intercept of these two lines. We use these two methods to estimate p_{c}. First, we use the CFL algorithm to generate 10 random lattices on two square areas of side length $L = 160$ and $L = 200$. Then we use the link random percolation process to generate 10^5 configurations for each of the five different values of p, $p = 0.332$, 0.333, 0.334, 0.335, and 0.336, and calculate $E_{\mathrm{p}}^{\mathrm{v+h}}(G(L), p)$ by averaging first over the 10^5 configurations on each lattice and then over the results of the 10 sample lattices. The resultant values are shown in figure 2. The first method yields $p_{\mathrm{c}} = 0.332\,98$ for both $L = 160$ and $L = 200$, and the second method gives $p_{\mathrm{c}} = 0.332\,94$. Note that the value of p_{c} may vary from one random lattice to another. For a lattice of size 160×160 or 200×200 and with the results of $E_{\mathrm{p}}^{\mathrm{v+h}}$ averaging over 10 sample lattices, we would expect the value of p_{c} to differ at the fifth digit from that obtained by taking the average over sufficiently large numbers of sample lattices. The error in the value of p_{c} caused by the calculation of $E_{\mathrm{p}}^{\mathrm{v+h}}$ can be roughly estimated as the difference between the values of p_{c} determined by the two methods. Therefore, the best estimate of the critical probability from our results is $p_{\mathrm{c}} = 0.3329(6)$.

To study the scaling functions, we use Hu's histogram Monte Carlo method to simulate bond percolation processes [26]. This method has been shown to be accurate and efficient, and it yields results as a continuous function of p. The latter is an important advantage for

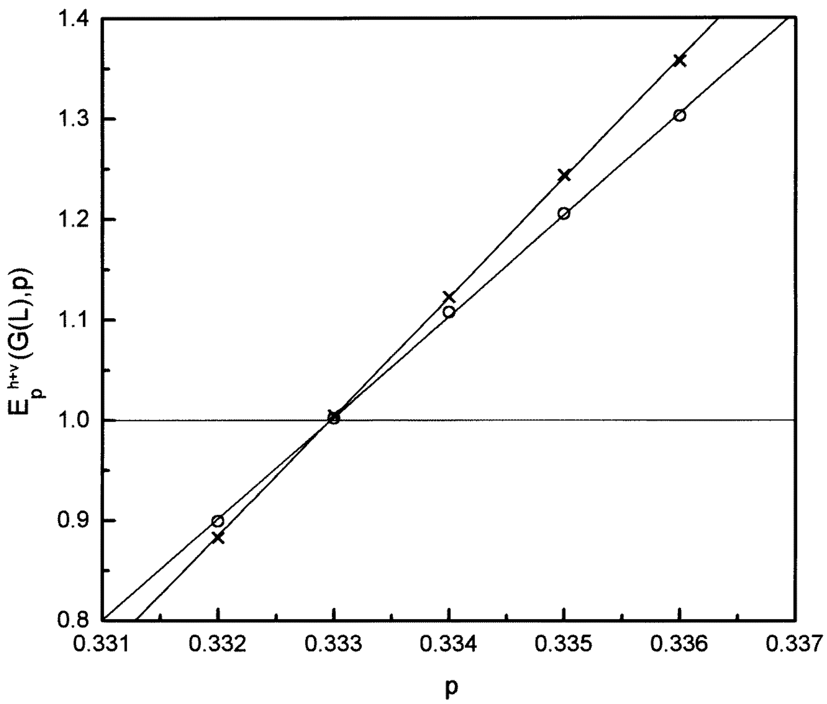

Figure 2. Numerical determination of p_c for bond percolation on random lattices. The results are obtained by averaging over the 10 sample random lattices for two different lattices sizes 160×160 (○) and 200×200 ($\times$).

the study of scaling functions. Here we briefly review the histogram Monte Carlo method and define related quantities. For a random lattice G with N sites and E links, first we choose w values of link occupied probabilities, $0 < p_1 < p_2 < \cdots < p_w < 1$. For a given p_j, $1 \leqslant j \leqslant w$, we use the link random percolation process to generate N_R different subgraphs. For each subgraph G', we calculate the total number of occupied links $b(G')$, the total number of clusters $n(G')$, and the total number of sites in percolating clusters $N^*(G')$. Then we use the above three quantities obtained from wN_R subgraphs to construct matrices: the total number of percolating subgraphs $N_{\mathrm{p}}(b, n)$, the total number of finite subgraphs $N_{\mathrm{f}}(b, n)$, and $N_{\mathrm{pp}}(b, n)$ which is the sum of $N^*(G')$, in terms of the number of occupied links b and the number of the clusters n. For a sufficiently large number of simulations, we can expect that the total number of percolating subgraphs with b occupied links and n clusters, $N_{\mathrm{tp}}(b, n)$, among the total 2^E subgraphs to be proportional to $N_{\mathrm{p}}(b, n)$,

$$N_{\mathrm{tp}}(b, n) = C(b) N_{\mathrm{p}}(b, n). \tag{4}$$

Similarly for $N_{\mathrm{tf}}(b, n)$, the total number of finite subgraphs with b occupied links and n clusters, we have

$$N_{\mathrm{tf}}(b, n) = C(b) N_{\mathrm{f}}(b, n). \tag{5}$$

Here the proportionality constant is

$$C(b) = \frac{C_b^E}{\sum_{n=1}^{N} (N_{\mathrm{p}}(b, n) + N_{\mathrm{f}}(b, n))}. \tag{6}$$

In terms of $N_{\rm tp}(b, n)$ and $N_{\rm tf}(b, n)$, we can write the partition function as

$$Z_N(G, p) = \sum_{b=0}^{E}\sum_{n=1}^{N}(N_{\rm tp}(b, n) + N_{\rm tf}(b, n))p^b(1-p)^{E-b}. \tag{7}$$

Then the percolating probability, $P(G, p)$, which is defined as the average value of the site number density contained in percolating subgraphs, can be written as

$$P(G, p) = \frac{1}{NZ_N(G, p)}\sum_{b=0}^{E}\sum_{n=1}^{N}N_{\rm tpp}(b, n)p^b(1-p)^{E-b} \tag{8}$$

with $N_{\rm tpp}(b, n) = C(b)N_{\rm pp}(b, n)$, and the existence probability, $E_{\rm p}(G, p)$, which is defined as the average number of percolating subgraphs, is in the form of

$$E_{\rm p}(G, p) = \frac{1}{Z_N(G, p)}\sum_{b=0}^{E}\sum_{n=1}^{N}N_{\rm tp}(b, n)p^b(1-p)^{E-b}. \tag{9}$$

The simulated data of $E_{\rm p}$ and P are used to study the corresponded scaling functions.

In the past few years the behaviour of $E_{\rm p}^{\rm v}(G, p)$ at the critical probability $p = p_{\rm c}$ for rectangular lattices of width a and height b with different aspect ratios $r = a/b$ and different boundary conditions has attracted researchers' interest. There is an unique value r_0 of r such that for the infinite system at the critical probability $p = p_{\rm c}$

$$E_{\rm p}^{\rm h}(r_0, p_{\rm c}) = E_{\rm p}^{\rm v}(r_0, p_{\rm c}). \tag{10}$$

This implies that in the infinite system we have

$$E_{\rm p}^{\rm h}(r_0, p_{\rm c}) = E_{\rm p}^{\rm v}(r_0, p_{\rm c}) = 0.5. \tag{11}$$

Because of the symmetry with respect to permutation of the two axes for a square lattice, one can expect that $r_0 = 1$. One can also use this argument to conclude that $r_0 = 1$ for random lattices, and this is confirmed by our Monte Carlo simulations. For other cases, it has been proposed that the value of r_0 is $\sqrt{3}/2$ for planar triangular lattices, and $\sqrt{3}$ for honeycomb lattices [25]. Ziff [10] pointed out that there is a finite-size correction to the value of r, but such a correction is not considered in this work.

Keeping $r = 1$, we first perform histogram Monte Carlo simulations on random lattices with $L = 8$, 14, and 30. In the simulations, the number of link occupied probabilities, w, is chosen to be about 185, and the number of subgraphs for a probability, N_R, is about 1.2×10^6. Then equations (8) and (9) are used to obtain P and $E_{\rm p}$, and the results are shown in figure 3. To investigate whether the obtained data exhibit the scaling behaviour of equation (1), we plot $E_{\rm p}$ and $P/L^{-\beta y_t}$ of three different sizes of random lattices as a function of $x = (p - p_{c)}L^{y_t}$ by using the exact values of $y_t = 3/4$ and $\beta = 5/36$ and the estimated value of the critical probability, $p_{\rm c} = 0.332\,96$. The results are shown in figure 4, and one can see that all the scaled data of $E_{\rm p}$ or P can be described by a single scaling function. Then we study the idea of universal scaling functions and non-universal metric factors by comparing the results of random lattices with those of square and planar triangular lattices with periodic boundary conditions. We choose the size of 30×30 for a square lattice, and 26×30 to approximate the aspect ratio $r = \sqrt{3}/2$ for a planar triangular lattice. For random lattices, we choose the size of 30×30, and average the results over 10 sample lattices. In the simulations, the parameters are chosen to be about $w = 250$ and $N_R = 1.2 \times 10^6$ for all the lattices. With the critical probability 0.5 for square lattices, 0.347 29 for planar triangular lattices [27], and 0.332 96 for random lattices, the simulated results and the corresponded scaled results of P and $E_{\rm p}$ are shown in figures 3 and 4. One

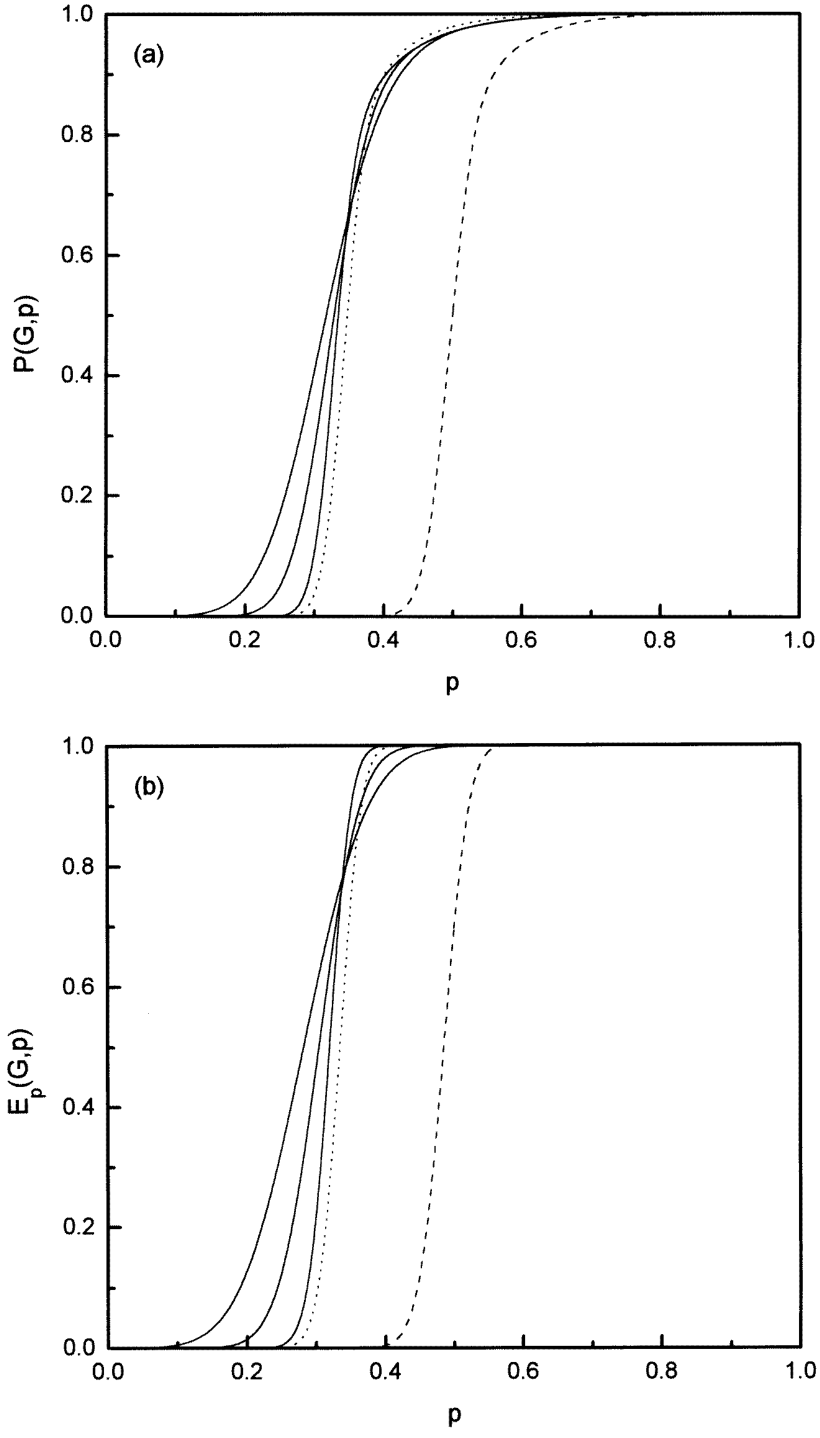

Figure 3. The results of (a) the percolating probabilities $P(G, p)$ and (b) the existence probabilities $E_{\rm p}(G, p)$ on random lattices (full curves), planar triangular lattices (dotted curve), and square lattices (dashed curve). The full curves below the intersections from left to right correspond to $L = 8$, $L = 14$, and $L = 30$.

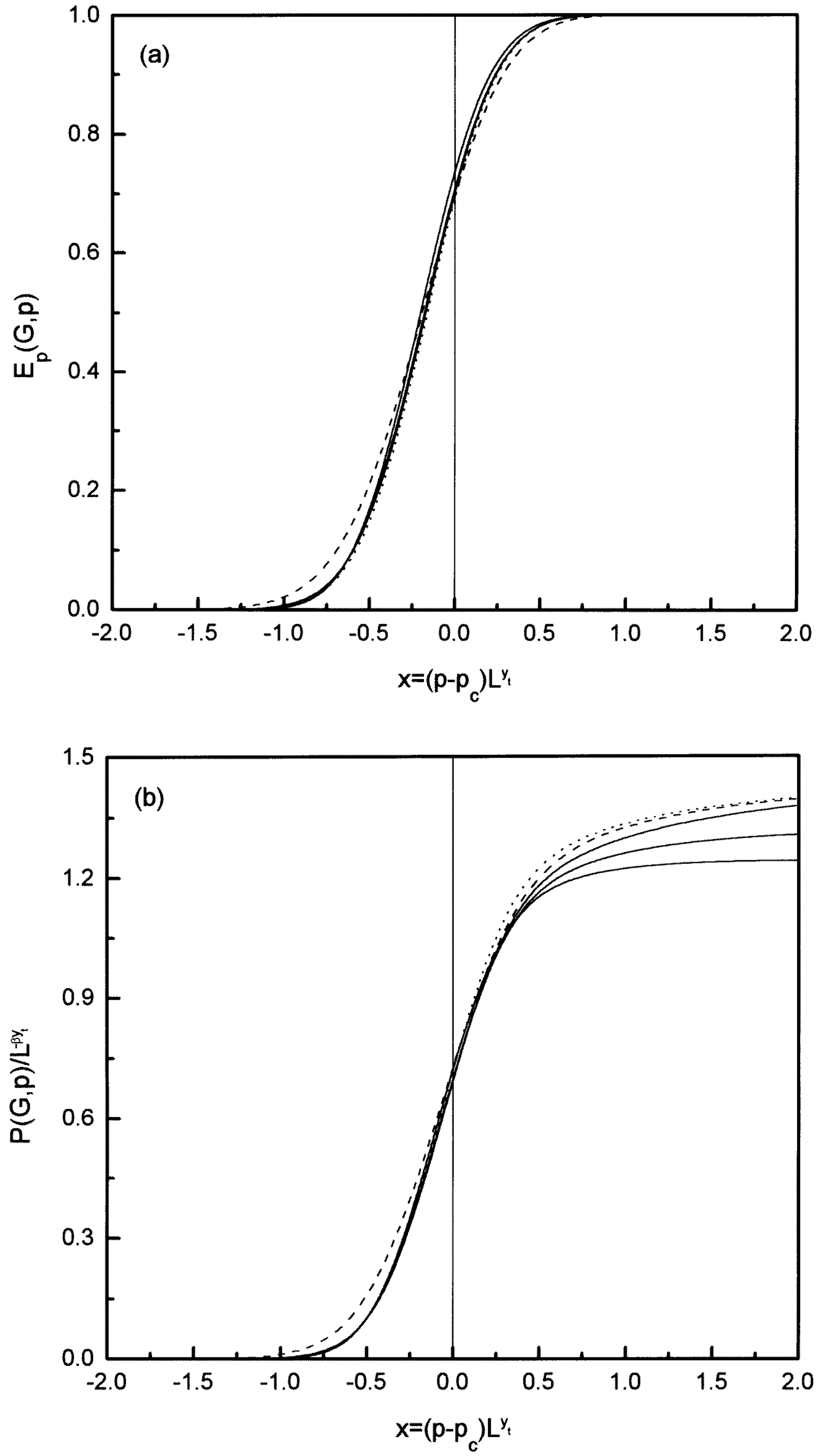

Figure 4. The scaled results of (a) E_p and (b) $P/L^{-\beta y_t}$ on random lattices (full curves), planar triangular lattices (dotted curve), and square lattices (dashed curve) as a function of $x = (p - p_\text{c})L^{y_t}$.

Table 1. The values of metric factors D_1, D_2, and D_3 for square, planar triangular, and random lattices with periodic boundary conditions.

Lattices	D_1	D_2	D_3
Square	1	1	1
Planar triangular	1.164(5)	1.174(9)	1.012(4)
	1.2(39)[a]	1.2(47)[a]	1.01(6)[a]
Random	1.160(0)	1.196(4)	1.065(6)

[a] From [19].

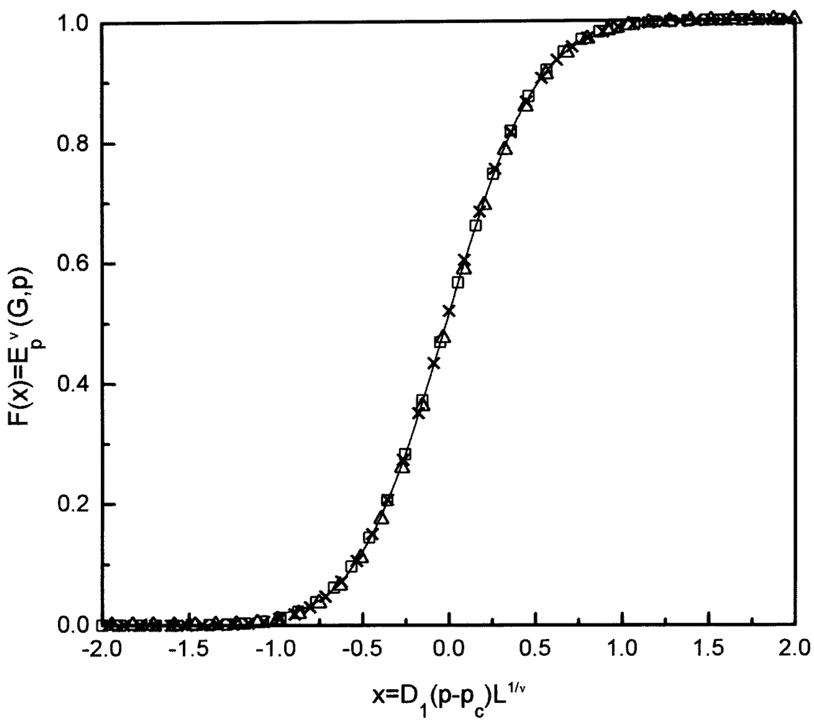

Figure 5. The scaling function $F(x)$ (full curve) with the data obtained from random lattices ($\times$), planar triangular lattices ($\triangle$), and square lattices ($\square$), where $x = D_1(p - p_c)L^{1/\nu}$.

can see from figure 4 that the scaled result of P or E_p for the planar triangular lattice are very close to those for random lattices.

Following HLC [19], we introduce three non-universal metric factors as follows. The first metric factor D_1 is introduced in E_p^v by the relation

$$E_p^v(G(L), p) = F(x) \tag{12}$$

with $x = D_1(p - p_c)L^{1/\nu}$. The other two metric factors, D_2 and D_3, are introduced in P by the relation

$$D_3 P(G(L), p) = L^{-\beta y_t} S(z) \tag{13}$$

with $z = D_2(p - p_c)L^{1/\nu}$. Note that E_p^v in the vicinity of p_c for an infinite lattice with aspect ratio r_0 takes the form

$$E_p^v(G(L), p) = 0.5 + A^v(p - p_c)L^{1/\nu} + B^v \tag{14}$$

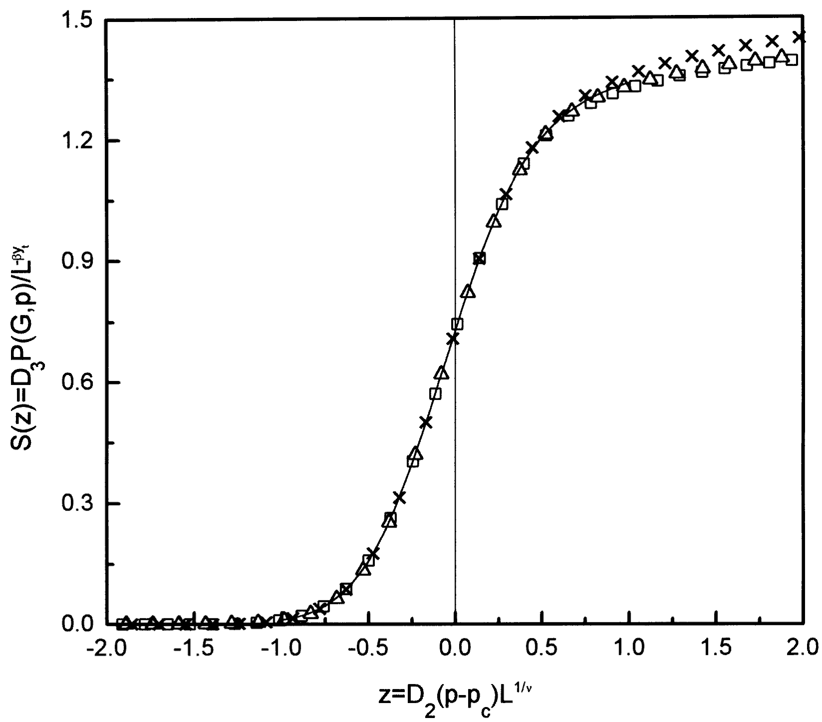

Figure 6. The scaling function $S(z)$ (full curve) with the data obtained from random lattices ($\times$), planar triangular lattices ($\triangle$), and square lattices ($\square$), where $z = D_2(p - p_c)L^{1/\nu}$.

where B^{v} is the order of $[(p - p_c)L^{1/\nu}]^2$. We use the value of A^{v} to approximate the value of D_1 for each lattice. The results of D_1 for square, planar triangular, and random lattices are given in table 1. Then we fit the data of $E_{\mathrm{p}}^{\mathrm{v}}$ as a polynomial of x up to the ninth power for each lattice, and take the average of the fittings of the three different types lattices as our final result. It yields the scaling function as

$$F(x) = 0.5149 + 0.9638x - 0.0584x^2 - 0.959(1)x^3 + 0.096(7)x^4 + \cdots.$$

This scaling function and the data are shown in figure 5. Note that if the coefficient of the linear term in the scaling function is normalized to 1, we have

$$F(\hat{x}) = 0.5149 + \hat{x} + k_2\hat{x}^2 + k_3\hat{x}^3 + k_4\hat{x}^4 + \cdots$$

with $k_{2,3,4} = -0.062(9)$, $-1.071(3)$, and $0.112(1)$. Comparing with the results $k_{2,3,4} = -0.517 \pm 0.010$, -1.08 ± 0.1, and 0.94 ± 0.15, obtained by Hovi and Aharony [21] two results agree in k_3 but disagree in k_2 and k_4. This may be due to the fact that we have used the periodic boundary condition on both directions and adopted the spanning rule of vertical crossing for a percolating cluster.

Similar procedures are applied to the percolating probability P. We use the factor D_3 to adjust the values of $S(z)$ such that $S(0)$ have the same value for the three different lattices, and the coefficients of the linear term in the power series of $(p - p_c)L^{1/\nu}$ are used to approximate the values of D_2. The values of D_2 and D_3 for different lattices are given in table 1, and the resultant scaling function is

$$S(z) = 0.726(4) + 1.341(5)z - 0.313(9)z^2 - 1.50(7)z^3 + 0.62(7)z^4 + \cdots$$

which is shown in figure 6 with the data. In table 1, we also give the values of D_1, D_2, and D_3 on a 433×500 planar triangular lattice obtained by HLC [19]. In comparison with their lattice, we have used a rather small lattice, 26×30, but the results are very close to each other.

In conclusion, we have located the critical probability of bond percolation on two-dimensional random lattices, and shown that the ideas of universal scaling functions and non-universal metric factors can be extended to random lattices. Our results show that, with the same scaling functions, the metric factors of random lattices agree with those of planar triangular lattices up to the first or the second digit. We speculate that after taking the average of the results over sufficiently large numbers of random lattices, the metric factors may be the same between random lattices and planar triangular lattices provided that the aspect ratios are 1 and $\sqrt{3}/2$.

Acknowledgments

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